Adventures in Scale Generation Along a Wilsonian Path, Part Two

by Warren Burt

In the first part of this article (in **1/1** 9:3), I discussed using the Partch/Novaro tonality diamond as a way of expanding the resources inherent in a combination product set ("CPS"). I also mentioned the technique of "stellation," explained by Paul Rapoport in his *Musicworks* article, as another expansion technique.

Since writing part one, I've been working with yet another technique of expanding the resources of combination product sets, and have developed a way of handling the some of the myriad of relationships inherent in this expansion.

The new technique of expansion is simplicity itself: it involves taking the members of a combination product set and making a new CPS out of them. For example, if you take five elements and make a 2/5 CPS out of them, you get a resulting set of ten elements. Now, take those ten elements and make a 2/10 CPS out of *them*. If all of your original five elements are not related by 2/1s, and are prime, you'll get a set of 45 elements, of which 35 are unique.

Rather than using numbers at the start of this project, I decided to use letters. I did this to see what relationships would result from these operations and to make a more "universal" kind of scale generator. That is, if the structure is represented by letters, you can substitute different numbers for the letters and then get different harmonies and scales, but the use of the letters will enable you to pick out parallel relationships between any two or more scales generated with this method. Also, please note that although I used the letters {a,b,c,d,e} as my symbols, I intend no correspondence whatever with the use of those letters to symbolize conventional musical pitches.

So, if we take five elements, {a,b,c,d,e}, and make a 2/5 CPS by multiplying every possible combination of two elements of the original set together, we get the ten-element set:

ab bc cd de
ac bd ce
ad be
ae

We now have a *dekany*—ten elements—to work with: {ab, ac, ad, ae, bc, bd, be, cd, ce, de}

Taking the ten elements of this dekany and making a 2/10 CPS with them, again by multiplying every possible combination of two elements from the original ten, we get the 45-element set:

abac acad adae aebc bcbd bdbe becd cdce cede
abad acae adbc aebd bcbe bdcd bece cdde
abae acbc adbd aebe bccd bdce bede
abbc acbd adbe aecd bcce bdde
abbd acbe adcd aece bcde
abbe accd adce aede
abcd acce adde
abce acde
abde

Thirty of these elements are unique in this set, but, since multiplication is commutative ($a \times b = b \times a$, for example), five elements occur more than once:

abcd = acbd = adbc
abce = acbe = aebc
abde = adbe = aebd
acde = adce = aedc
bcde = bdce = becd

This gives a total of 35 elements in the resulting set. Further, each of the 30 unique elements has one of the original five factors (of our 2/5 CPS) occurring twice within itself. These elements can be grouped by which of the original five elements appear in them:

aa	bb	cc	dd	ee
abac	abbc	accd	adde	aebe
abad	abbd	acce	bdde	aece
abae	abbe	acbc	cdde	aede
acad	bcbd	bccd	adbd	bece
acae	bcbe	bcce	adcd	bede
adae	bdbe	cdce	bdcd	cede

If the elements that make up the original five-element set are low prime numbers, the six-note chords resulting from these multiplications will all have some sort of fairly consonant harmonic relationships.

(I use lists of letters here to show relationships. Erv Wilson's very beautiful geometrical diagrams and three-dimensional models do the same thing. My thinking, though, is more biased to lists than to visual models, even if very beautiful, so I chose to use my "alphanumeric" notation as a way in which I could see relationships easily.)

Further, if we take any of the factors of the dekany and list all the elements of the 35-element set that contain that original factor, we get, for example, for the dekany element "ab":

abac acbc adbc aebc
abad acbd adbd aebd
abae acbe adbe aebe
abbc
abbd
abbe
abcd
abce
abde

Again, some of these elements are duplicated:

abcd = acbd = adbc
abce = acbe = aebc
abde = adbe = aebd

This gives us a set of twelve unique elements that have "ab" as their common factor.

Again, if the original five elements are all low primes, these twelve elements will all be fairly closely related harmonically.

Therefore, you could extract ten unique twelve-element sets (one for each of the factors of the original dekany), five six-element sets (with one of the original factors duplicated), and one five-element set (of all the possible combinations of four elements of the original five) out of this 35-element set.

Further, each of the twelve-element sets could be broken up into smaller units ("chords") by extracting those elements that contain common factors. For example, for the "ab" set listed above:

Factor:	a	b	c	d	e
	all 12	all 12	abac	abad	abae
			abbc	abbd	abbe
			abcd	abcd	abce
			abce	abde	abde
			acbc	adbd	aebe

And taking elements where either a or b occur twice:

aa	bb
abac	abbc
abad	abbd
abae	abbe

We now have our twelve-element set broken up into three five-element subsets and two three-element subsets. Further, each of these subsets has at least one element in common with another subset. If low prime numbers are used for the source set, the resulting numerical relationships begin to have strong resemblances to a traditional harmonic system. (For example, an F major 9th chord has three tones in common with a C major 9th chord. This C major 9th chord has three tones in common with a G major 9th chord, and so on.)

Finally, each of the five six-element sets (containing two occurrences of the same original factor) can be divided up into four three-element subsets, again by grouping elements by their common factor. For example, for the set made from aa: {abac, abad, abae, acad, acae, adae} the following three-element subsets can be extracted:

Factor:	b	c	d	e
	abac	abac	abad	abae
	abad	acad	acad	acae
	abae	acae	adae	adae

These four subsets are even more tightly related than the five-element subsets of the twelve-element sets. Here, each element of the "b" subset, for example, occurs in one of the other three subsets. And again, if low primes are used for the factors, the results will be a very simple harmonic system of three-note chords nested within a six-note scale.

I've kept this all abstract up to this point, because I wanted to see what relationships would result, regardless of which numbers I used to make my eventual sets. As stated, if low prime numbers are used, the results will have some sounds in common with traditional harmony. But if higher primes are used, we can still use these relationships to assemble chords that, although not traditional sounding, will be numerically related, and may, perhaps, function in their scales in a manner analogous to that in which the more harmonic sounding chords function in theirs.

In Part 1, I expressed the desire to make a work that explored the internal relationships of the dekanies described therein in a more rigorous way than I had done in "Fugue" and "Preludes." Although I was not working with the dekanies of Part 1, using this kind of set and subset structure gave me a chance to observe the results of the kind of extraction of internal relationships this structure implied.

I decided to take three sets of five numbers and make scales using this method. Then, I wrote a three movement piece that used these resulting sets. Each movement is exactly the same as the other two, with the sole exception of using the different pitches of the different scales. That is, the pitch that resulted from, say, "abcd" from the first set of five numbers in the first movement is replaced by the pitch that resulted from "abcd" from the second set of five numbers in the second movement; this, in turn, is replaced by the pitch that resulted from "abcd" from the third set of five numbers in the third movement. In this way, I could hear the results of the different harmonic worlds generated by different original sets of numbers in a very clear and immediate way.

I decided to use the harmonic progressions inherent in the twelve-note "scales" as the basis for the harmony of the piece. The order of the chords in the presentation of each scale is determined by common tones in successive chords. The chords for the scale made of notes with "ab" as a factor are shown in Table 1.

The ordering chosen was e, aa, d, bb, c.

Chord e has note "abae" in common with chord aa. Chord aa has note "abad" in common with chord d. Chord d has note "abbd" in common with chord bb. Chord bb has note "abbc" in common with chord c.

Further, each five-note chord occurs identically in more than one scale. This provides a way to link up progressions in various scales through the use of common chords. For example, chord c in scale "ab" is identical with chord a of scale "bc." The chords for scale "bc" appear in Table 2.

In this way, I devised a progression of chords and

Table 1. Chords for Scale "ab"

	Three five-note chords			Two three-note chords with two of a factor	
Factor:	c	d	e	aa	bb
	abac	abad	abae	abac	abbc
	abbc	abbd	abbe	abad	abbd
	abcd	abcd	abce	abae	abbe
	abce	abde	abde		
	acbc	adbd	aebe		

Table 2a. Chords for Scale "bc"

	Three five-note chords			Two three-note chords with two of a factor	
Factor:	a	d	e	bb	cc
	abac	abcd	abce	abbc	acbc
	abbc	bcbd	bcbe	bcbd	bccd
	abcd	bccd	bcce	bcbe	bcce
	abce	bcde	bcde		
	acbc	bdcd	bece		

Table 2b. Common Chords

Chord of scale "ab"	abac	Chord a of scale "bc"	abac
C	abbc		abbc
	abcd		abcd
	abce		abce
	acbc		acbc

scales that included all of the possible chords in all of the ten possible twelve-note scales generated with this method, each chord being related to the next by at least one common tone. After all the scales were thus traversed, a coda was added that had each of the five six-note subsets (of notes with a factor occurring twice) articulated by a "common-tone" ordering of their four three-note chords, and a final measure that used the one five-note subset (of all the ways of combining four of the original five elements) as an ending chord.

Once this progression was defined, the choosing of pitch orders, octave placements, durations, and loudnesses could occur freely, with the knowledge that a melodic contour chosen in one tuning might be altered when the notes from a second tuning were applied to it. That is, in one tuning, a succession of the notes represented by "abcd," "acad," and "accd" might go up, whereas in another tuning, a progression of notes represented by the same factors might go down.

To make the scales to use in this piece, I decided to use three sets of five numbers that gradually ascended

Table 3. Working out the 2/5 CPS dekany of {3,5,11,17,31} (a = 3; b = 5; c = 11; d = 17; e = 31)

ab = 15	bc = 55	cd = 187	de = 527
ac = 33	bd = 85	ce = 341	
ad = 51	be = 155		
ae = 93			

Table 4. The 2/10 CPS of the dekany in Table 3

abac = 495	acad = 1683	adae = 4743	aebc* = 5115	bcbd = 4675
abad = 765	acae = 3069	adbc* = 2805	aebd* = 7905	bcbe = 8525
abae = 1395	acbc = 1815	adbd = 4335	aebe = 14415	bccd = 10285
abbc = 825	acbd* = 2805	adbe* = 7905	aecd* = 17391	bcce = 18755
abbd = 1275	acbe* = 5115	adcd = 9537	aece = 31713	bcde* = 28985
abbe = 2325	accd = 6171	adce* = 17391	aede = 49011	
abcd* = 2805	acce = 11253	adde = 26877		
abce* = 5115	acde* = 17391			
abde* = 7905				
bdbe = 13175	becd* = 28985	cdce = 63767	cede = 179707	
bdcd = 15895	bece = 52855	cdde = 98549		
bdce* = 28985	bede = 81685			
bdde = 44795				

* = duplicated members of the set

the prime-number series. Partly to see what would happen, partly out of perversity, my lowest set had elements related by a 2:1 in it, whereas the other two sets were all prime numbers.

Set 1: {2, 3, 4, 5, 7}; Set 2: {1, 3, 5, 7, 11}; Set 3: {3, 5, 11, 17, 31} (second, third, fifth, seventh, and eleventh primes).

Because of octave duplications, the scale made from the 2/10 CPS of the 2/5 CPS of set 1 had only 19 notes. The use of 7 as the highest prime number and the presence of 2 and 4 (both doublings of 1) gave the resulting scale a strongly tonal, harmonic, 7-limit feeling.

The scales based on sets 2 and 3 were both 35-note scales, as all the factors were prime, but the presence of 1 in set 2 meant that ratios of 11-limit and lower were given a chance to be heard. For example three pitches made of the products of {1, 1, 7, 11}, {1, 1, 5, 11} and {1, 1, 3, 11} are related as 7:5:3, a fairly consonant chord.

With 3 as the lowest factor of set 3, and 31 as the highest factor, harmonies based on higher prime limits could be heard in the third scale, but the presence of the 3 and 5 guaranteed that at least some of the relationships heard would be related to traditional tonal harmonies.

In the end though, I was surprised by just *how* tonal the results of these scales were. In the piece, called "3 Cat Laxative Sonatas" (and how the piece got its name is another story entirely, one mercifully outside the area of interest of a music theory journal[1]), the first movement sounds positively Lisztian in places, while the substitution of the 11-limit source set scale in the second movement has a more "dissonant," but still strongly tonal feel. The surprise is the third movement, which uses the set {3, 5, 11, 17, 31} as its source. Here, rather than being simply dissonant, or "atonal," the harmonies (to my ears, anyway) still sound strongly related, and remind me of harmonies used by Scriabin.

Here is how I worked out one of the scales. Table 3 is the multiplication of the original factors to make the 2/5 CPS; Table 4 shows the multiplications of the 2/10 CPS. Table 5 (p. 8) shows the cents values of the scale based on the prime number 3 being equal to 1/1; and Table 6 (p. 9) lists the scale in ascending order with the way I implemented it using five MIDI channels of my Roland SCC-1 synthesizer.

The other two scales were worked out in the same way. The other two scales, with their lower factors, did not have the plethora of small intervals that exist in this scale. Here are the other two scales. Table 7 (p. 9) gives the scale resulting from the source set {1, 3, 5, 7, 11}, and Table 8 (p. 10) the scale made from {2, 3, 4, 5, 7}. Again, the tables include the MIDI implementation for my SCC-1. (Some might object to the SCC-1's way of implementing microtonality (multiple MIDI channels for a scale more than twelve notes) but I have found that with appropriate software that allows one to link a note number and a MIDI channel (such as Russ Kozerski's "Drummer" or John Dunn's "Kinetic Music Machine") it works just fine.

Table 5. Cents values of the scale in Table 2 with 3 = 1/1				
abac = 440	acad = 158	adae = 752	aebc* = 883	bcbd = 727
abad = 1193	acae = 1198	adbc* = 1043	aebd* = 436	bcbe = 567
abae = 1033	acbc = 289	adbd = 596	aebe = 276	bccd = 892
abbc = 124	acbd* = 1043	adbe* = 436	aecd* = 601	bcce = 732
abbd = 878	acbe* = 883	adcd = 761	aece = 441	bcde* = 286
abbe = 718	accd = 8	adce* = 601	aede = 1195	
abcd* = 1043	acce = 1048	adde = 155		
abce* = 883	acde* = 601			
abde* = 436				
bdbe = 121	becd* = 286	cdce = 451	cede = 1044	
bdcd = 446	bece = 126	cdde = 4		
bdce* = 286	bede = 880			
bdde = 1039				
* = duplicated members of the set				

Note that in in the ninteen-tone scale in Table 8, many products can result in the same pitch. The "tonal weighting" inherent in such a scheme is especially evident in this scale.

For purposes of comparison, here are two chords as they exist in all three scales:

Scale "ab" chord e: cents comparisons

Factors	31-limit	11-limit	7-limit
abae	1033	53	471
abbe	718	755	1173
abce	883	440	471
abde	436	1022	857
aebe	276	605	240

Scale "ab" chord aa: cents comparisons

Factors	31-limit	11-limit	7-limit
abac	440	1088	702
abad	1193	471	1088
abae	718	53	471

Note that in the 7-limit chord e, two of the pitches are the same. A potential five-note chord collapses to a four-note chord. Note also that the chord aa expresses the ratio c:d:e, allowing for the common factors "aba" in all three elements. This means that in the 7-limit scale, this is a 4:5:7 chord, whereas in the 11-limit scale, it's a 5:7:11 chord, and in the 31-limit scale, it's a still fairly consonant 11:17:31 chord. In fact, in the more dissonant context of the harmony of the 31-limit movement, the relative consonance of this 11:17:31 chord comes as quite a shock.

One other note: my cents values for the 31-limit scale were based on using 3 as my 1/1. For the 11-limit and 7-limit scales, however, I used 1 as my 1/1. In effect, this means that the 31-limit scale is a 3/2 above the other two scales. But even though this might be a real numerical relationship, in the context of this piece, I don't think the different transpositions of the scales are hearable. My ear, at any rate, is more immediately grabbed by the distinct harmonic worlds of the three scales, and is fascinated by the contrast between the harmonic "flavor" of each scale, rather than by any larger transpositional relationships between the scales.

The results of this work satisfied and surprised me greatly. After the extreme dissonance of "*Vingt Enflures sur L'enfant Melvin*" (see Part One), the tonal 7-limit harmonies were quite shocking. I laughed as I realized I was writing a piece that was an unashamed example of "neo-Romanticism" in spite my nonromantic exploratory intentions. I was also pleased with the way the differences between the scales were clearly hearable from movement to movement. And the piece, for me, works not only on a didactic level, but also a musical one: it's fun to hear, as well as being another exploration of the rich harmonic universe implied by the work of Erv Wilson. **1/1**

Note:

1. Okay, okay, for those of you who can't live without tabloid journalism (or is that tabloid musicology?): While writing this piece, I was visiting my composer friend Janika

Table 6.
35-note scale made from 2/10 CPS of dekany of 2/5 CPS of {3,5,11,17,31} with SCC1 setup: (3 = 1/1 = 0 cents)

Degree	Factors	Cents	Key	MIDI Ch.	SCC-1 Tuning
1	cdde	4	C	1	C +4
2	accd	8	C	2	C +8
3	bdbe	121	C♯	1	C♯ +21
4	abbc	124	C♯	2/3	C♯ +24
5	bece	126	C♯	4/5	C♯ +26
6	adde	155	D	1/2/3	D –45
7	acad	158	D	4/5	D –43
8	aebe	276	D♯	1/2	D♯ –24
9	bcde	286	D♯	3/4	D♯ –14
10	acbc	289	D♯	5	D♯ –11
11	abde	436	E	1/2/3	E +36
12	abac	440	E	4/5	E +40
13	aece	441	F	1/2	F –59
14	bdcd	446	F	3/4	F –54
15	cdce	451	F	5	F –49
16	bcbe	567	F♯	1/2	F♯ –33
17	adbd	596	F♯	3/4	F♯ –4
18	acde	601	F♯	5	F♯ +1
19	abbe	718	G	1/2	G +18
20	bcbd	727	G	3/4	G +27
21	bcce	732	G	5	G +32
22	adae	752	G♯	1/2/3	G♯ –48
23	adcd	761	G♯	4/5	G♯ –39
24	abbd	878	A	1/2	A –22
25	bede	880	A	3	A –20
26	abce	883	A	4	A –17
27	bccd	892	A	5	A –8
28	abae	1033	A♯	1/2	A♯ +33
29	bdde	1039	A♯	3/4	A♯ +39
30	abcd	1043	A♯	5	A♯ +43
31	cede	1044	B	1/2/3	B –56
32	acce	1048	B	4/5	B –52
33	abad	1193	C	3	C –7
34	aede	1195	C	4	C –5
35	acae	1198	C	5	C –2

Table 7.
35-note scale made from 2/10 CPS of dekany of 2/5 CPS of {1,3,5,7,11} with SCC-1 setup: (1 = 1/1 = 0 cents)

Degree	Factors	Cents	Key	MIDI Ch.	SCC–1 Tuning
1	bccd	43	C	12	C +43
2	abae	53	C	13/14	C +53
3	cede	58	C♯	11	C♯ –42
4	adde	89	C♯	12	C♯ –11
5	acce	124	C♯	13/14	C♯ +24
6	acad	155	D	11	D –45
7	bcde	208	D	12	D +8
8	adbd	240	D	13/14	D +40
9	acbc	275	D♯	11	D♯ –25
10	aece	289	D♯	12	D♯ –11
11	adae	320	D♯	13/14	D♯ +20
12	bcbd	359	E	11	E –41
13	bede	373	E	12	E –27
14	abce	440	E	13/14	E +40
15	abad	471	F	11	F –29
16	cdde	475	F	12	F –25
17	bdbe	524	F	13	F +24
18	accd	541	F	14	F +41
19	abbc	590	F♯	11	F♯ –10
20	aebe	605	F♯	12	F♯ +5
21	bdcd	626	F♯	13/14	F♯ +26
22	acde	706	G	11/12	G +6
23	abbe	755	G	13/14	G +55
24	bdde	791	G♯	11	G♯ –9
25	bcce	826	G♯	12	G♯ +26
26	abcd	857	G♯	13/14	G♯ +57
27	aede	871	A	11/12	A –29
28	acae	938	A	13/14	A +38
29	bece	991	A♯	11/12	A♯ –9
30	abde	1022	A♯	13/14	A♯ +22
31	abac	1088	B	11	B –12
32	cdce	1093	B	12	B –7
33	adcd	1124	B	13	B +24
34	bcbe	1142	B	14	B +42
35	abbd	1173	C	11	C –27

Table 8. 19-note scale made from 2/10 CPS of dekany of 2/5 CPS of {2,3,4,5,7} with SCC-1 set up: (1 = 1/1 = 0 cents)					
Degree	Factors	Cents	Key	MIDI Ch.	SCC–1 Tuning
1	bdde	43	C	8	C +43
2	acde adae cdce	155	C♯	7/8	C♯ +55
3	abbc	204	D	7	D +4
4	aebe bece	240	D	8	D +40
5	adbd bdcd	275	D♯	7/8	D♯ –25
6	bdbe	359	E	7	E –41
7	acad accd	386	E	8	E –14
8	abae abce bcce	471	F	7	F –29
9	adde cdde	541	F	8	F +41
10	abbd bcbd	590	F♯	7	F♯ –10
11	bede	626	F♯	8	F♯ +26
12	abac acbc	702	G	7	G +2
13	aece	738	G	8	G +38
14	adcd	773	G♯	7/8	G♯ –27
15	abde bcde	857	A	7/8	A –43
16	acae	969	A♯	7/8	A♯ –31
17	abad abcd acce bccd	1088	B	7	B –12
18	aede cede	1124	B	8	B +24
19	abbe bcbe	1173	C	7	C –27

Vandervelde, who lives in St. Paul, Minnesota with her two cats, White Kitty and Pumpkin Kitty. The cats were having hairball problems. So she was giving them this "cat laxative" to help cure the problem. Only thing was, she would leave the tube of cat laxative on the bathroom sink, next to the toothpaste tube. Though no human, to my knowledge, has ever brushed her teeth with the cat laxative, and no kitty has ever had "mint-fresh breath" as a result of a reciprocal confusion, the juxtaposition was just a bit too precious for my (admittedly adolescent) sensibilities. Each of the movements of the piece is named after one of the inhabitants of the house: first movement (7-limit) is "White Kitty." second movement (11-limit) is "Pumpkin Kitty." And the third (31-limit) is "Janika." Listeners may draw whatever conclusions they wish from these harmonic/nominal juxtapositions.